

# SPECTRAL PROBLEMS FOR NON ELLIPTIC SYMMETRIC SYSTEMS WITH DISSIPATIVE BOUNDARY CONDITIONS

FERRUCCIO COLOMBINI, VESSELIN PETKOV AND JEFFREY RAUCH

**ABSTRACT.** This paper considers and extends spectral and scattering theory to dissipative symmetric systems that may have zero speeds and in particular to strictly dissipative boundary conditions for Maxwell's equations. Consider symmetric systems  $\partial_t - \sum_{j=1}^r A_j \partial_{x_j}$  in  $\mathbb{R}^n$ ,  $n \geq 3$ ,  $n$  odd, in a smooth connected exterior domain  $\Omega := \mathbb{R}^n \setminus \bar{K}$ . Assume that the rank of  $A(\xi) = \sum_{j=1}^n A_j \xi_j$  is constant for  $\xi \neq 0$ . For maximally dissipative boundary conditions on  $\Omega := \mathbb{R}^n \setminus \bar{K}$  with bounded open domain  $K$  the solution of the boundary problem in  $\mathbb{R}^+ \times \Omega$  is described by a contraction semigroup  $V(t) = e^{tG_b}$ ,  $t \geq 0$ . Assuming coercive conditions for  $G_b$  and its adjoint  $G_b^*$  on the complement of their kernels, we prove that the spectrum of  $G_b$  in the open half plane  $\operatorname{Re} z < 0$  is formed only by isolated eigenvalues with finite multiplicities.

**2000 Mathematics Subject Classification:** Primary 35P25, Secondary 47A40, 35L50, 91U40

**Keywords:** non elliptic symmetric systems, dissipative boundary conditions, asymptotically disappearing solutions

## 1. INTRODUCTION

This paper is devoted to dissipative symmetric hyperbolic systems on exterior domains. The systems studied have two properties that render their study difficult. First, they are dissipative and not conservative so self adjointness techniques are not available. Second, the symbol  $A(\xi)$  is not elliptic. However, we suppose that the failure of ellipticity is uniform in  $\xi \neq 0$  and that coercivity estimates hold on the orthogonal complement of the space  $H_b$  spanned by the stationary solutions. We present new techniques that suffice to show that the spectrum of the generator  $G_b$  restricted to  $H_b^\perp$  consists of the imaginary axis plus at most discrete subset of  $\operatorname{Re} z < 0$  and that the discrete part is stable under perturbations. For the wave equation with dissipative boundary conditions such results have been proved before (see [10]).

Suppose that  $n \geq 3$  is odd, and that  $K \subset \mathbb{R}^n$  is a bounded open set with smooth boundary. Assume that the exterior domain  $\Omega := \mathbb{R}^n \setminus \bar{K}$  is connected. Choose  $\rho > 0$  so that  $K \subset \{x \in \mathbb{R}^n : |x| \leq \rho\}$ . Suppose that for  $j = 1, \dots, n$ ,  $A_j$  are

---

JR was partially supported by NSF grant DMS 0807600 .

VP was partially supported by ANR project Nosevol BS01019 01.

$(r \times r)$  hermitian symmetric matrices and set  $A(\xi) = \sum_{j=1}^n A_j \xi_j$ . Assume that  $\text{Rank } A(\xi) = r - d_0 > 0$  is independent of  $\xi \neq 0$ . Define  $G := \sum_{j=1}^n A_j \partial_{x_j}$  and for  $x \in \partial\Omega$ , denote by  $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$  the unit outward normal to  $\Omega$ , and  $A(\nu(x)) = \sum_{j=1}^n A_j \nu_j(x)$ .

Suppose that  $\mathcal{N}(x) \subset \mathbb{C}^r$  is a linear space depending smoothly on  $x \in \partial\Omega$  such that

- (i)  $\langle A(\nu(x))u(x), u(x) \rangle \leq 0$  for all  $u(x) \in \mathcal{N}(x)$ ,
- (ii)  $\mathcal{N}(x)$  is maximal with respect to (i).

Then the initial boundary value problem

$$(1.1) \quad \begin{cases} (\partial_t - G)u = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u(t, x) \in \mathcal{N}(x) & \text{for } t \geq 0, \quad x \in \partial\Omega, \\ u(0, x) = f(x) & \text{in } \Omega \end{cases}$$

yields a contraction semigroup  $V(t) = e^{tG_b}$ ,  $t \geq 0$ , in  $\mathcal{H} = L^2(\Omega : \mathbb{C}^r)$  with generator  $G_b$ . The domain  $D(G_b)$  of the generator is the closure with respect to the graph norm  $(\|g\|^2 + \|Gg\|^2)^{1/2}$  of functions  $g(x) \in C_{(0)}^1(\bar{\Omega} : \mathbb{C}^r)$  satisfying the boundary condition  $g(x)|_{\partial\Omega} \in \mathcal{N}(x)$ .

Introduce the unitary group  $U_0(t) = e^{tG_0}$  on  $H_0 = L^2(\mathbb{R}^n : \mathbb{C}^r)$  solving the Cauchy problem

$$(1.2) \quad \begin{cases} (\partial_t - G)u = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ u(0, x) = f(x) & \text{in } \mathbb{R}^n. \end{cases}$$

Define  $\mathcal{H}_b \subset \mathcal{H}$  to be the space generated by the eigenvectors of  $G_b$  with eigenvalues  $\mu \in i\mathbb{R}$  and let  $\mathcal{H}_b^\perp$  be the orthogonal complement of  $\mathcal{H}_b$  in  $\mathcal{H}$ . The generator  $G_0 = \sum_{j=1}^n A_j \partial_{x_j}$  is skew self-adjoint in  $H_0$  and the spectrum of  $G_0$  on the space  $H_0^{ac} = (\text{Ker } G_0)^\perp \subset H_0$  is absolutely continuous (see Chapter IV in [13]).

Throughout this paper we suppose that  $G_b$  and  $G_b^*$  satisfy the following coercive estimates

$(H)$  : For each  $f \in D(G_b) \cap (\text{Ker } G_b)^\perp$  we have

$$(1.3) \quad \sum_{j=1}^n \|\partial_{x_j} f\| \leq C(\|f\| + \|G_b f\|)$$

with a constant  $C > 0$  independent of  $f$ .

$(H)^*$  : For each  $f \in D(G_b^*) \cap (\text{Ker } G_b^*)^\perp$  we have

$$(1.4) \quad \sum_{j=1}^n \|\partial_{x_j} f\| \leq C(\|f\| + \|G_b^* f\|)$$

with a constant  $C > 0$  independent of  $f$ .

Notice that when  $G$  is elliptic, that is  $d_0 = 0$ , these estimates hold exactly when the associated boundary value problems satisfy Lopatinski conditions. The

interesting cases are when  $G$  is not elliptic and the estimates only hold on the orthogonal complement to the kernels. Majda [7] proved  $(H)$  and  $(H^*)$  when the constant rank hypothesis holds and in addition the following condition is satisfied.

$(E)$  : There exists a first order  $(l \times r)$  matrix operator  $Q = \sum_{j=1}^n Q_j \partial_{x_j}$  so that

$$Q(\xi)A(\xi) = 0, \quad \text{Ker } Q(\xi) = \text{Range } A(\xi),$$

where  $Q(\xi) = \sum_{j=1}^n Q_j \xi_j$ .

This says that one has an exact sequence

$$\mathbb{C}^r \xrightarrow{A(\xi)} \mathbb{C}^r \xrightarrow{Q(\xi)} \mathbb{C}^r.$$

When the kernel of  $A(\xi)$  has constant rank it is known that we can always choose a polynomial with matrix coefficient  $Q(\xi)$  for which the above sequence is exact (see Section 4). On the other hand, if  $(E)$  holds, the symbol  $Q(\xi)$  is a linear function of  $\xi$  and the second order matrix operator  $(-G^2 + Q^*Q)$  is strongly elliptic.

**Example 1.1.** Our principal motivation is the Maxwell equations

$$\partial_t E - \text{curl } B = 0, \quad \partial_t B + \text{curl } E = 0, \quad \text{div } E = 0, \quad \text{div } B = 0.$$

For this system one can take

$$Q \begin{pmatrix} E \\ B \end{pmatrix} := \begin{pmatrix} \text{div } E \\ \text{div } B \end{pmatrix}.$$

Majda [7] proves that the Maxwell system with strictly dissipative boundary conditions satisfies hypothesis  $(H)$  and  $(H)^*$ .

For dissipative symmetric systems some solutions can have global energy decreasing exponentially as  $t \rightarrow \infty$  and it is possible also to have disappearing solutions. The precise definitions are the following.

**Definition 1.2.** The solution  $u := V(t)f$  is a disappearing solution (DS), if there exists  $T > 0$  such that  $V(t)f = 0$  for  $t \geq T$ .

**Definition 1.3.** The solution  $u := V(t)f$  is an asymptotically disappearing solution (ADS), if there exists  $\lambda \in \mathbb{C}$  with  $\text{Re } \lambda < 0$  and  $f \neq 0$  such that  $V(t)f = e^{\lambda t}f$ .

An (ADS) is generated by an eigenfunction  $G_b f = \lambda f$  with  $f \neq 0$  and  $\text{Re } \lambda < 0$ . The existence of at *least one* (ADS) implies the non completeness of the wave operators  $W_{\pm}$ , that is  $\text{Ran } W_- \neq \text{Ran } W_+$  (see [14] and Section 4). The disappearing solutions perturb considerably the inverse back-scattering problems related to the leading singularity of the scattering kernel (for more details see [14]). Disappearing solutions for the wave equation have been studied by Majda [8] and for the symmetric systems by Georgiev [4]. On the other hand, (ADS) for Maxwell's equations with dissipative boundary conditions on the sphere  $|x| = 1$  are constructed in [1]. One motivation for the present work is to show that the latter (ADS) persist under small perturbations. To show that we examine the spectrum of  $G_b$  in the half plane  $\text{Re } z < 0$ . The main result is the following

**Theorem 1.4.** *Assume the hypothesis  $(H)$ ,  $(H)^*$  fulfilled. Then the spectrum of  $G_b$  restricted to  $\mathcal{H}_b^{\perp}$  is formed by a discrete set in  $\{z \in \mathbb{C} : \text{Re } z < 0\}$  of eigenvalues with finite multiplicities and a continuous spectrum on  $i\mathbb{R}$ .*

A similar result has been proved in [10] for the semigroup  $V_w(t) = e^{tG_w}$ ,  $t \geq 0$ , related to the mixed problem

$$(1.5) \quad \begin{cases} (\partial_t^2 - \Delta)w = 0 \text{ in } \mathbb{R}^+ \times \Omega, \\ \partial_\nu w - \alpha(x)\partial_t w = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega, \\ (w(0, x), w_t(0, x)) = (f_1, f_2), \end{cases}$$

where  $\alpha \geq 0$  is a smooth function on  $\partial\Omega$ . Let  $V_0(t) = e^{tG_f}$  be the unitary group related to the Cauchy problem for the wave equation in  $\mathbb{R} \times \mathbb{R}^n$ . The strategy of Lax-Phillips in [10] was to prove that for  $\operatorname{Re} \lambda > 0$  the difference

$$(G_w - z)^{-1} - (G_f - z)^{-1}$$

is a compact operator.

For symmetric systems whose generators are not elliptic and for which coercivity holds only for data satisfying additional constraints, like the divergence free constraints for the Maxwell equations, the coercivity estimates for the generators hold on different spaces. This prevents a direct application of the strategy in [10]. We show that for  $\operatorname{Re} z < 0$  the operator  $G_b - z$  on  $\mathcal{H}_b^\perp$  is Fredholm. The main difficulty is to prove that there exists at least one point  $z_0$ ,  $\operatorname{Re} z_0 < 0$ , such that the index of  $G_b - z_0$  is equal to zero. Our proof uses scattering theory. We apply the representation of the scattering matrix  $\mathcal{S}(z)$  established for elliptic systems with conservative boundary conditions in [11] and for non elliptic systems with dissipative boundary conditions in [14].

The paper is organized as follows. In Section 2 we study the Fredholm operator  $G_b - z$ ,  $\operatorname{Re} z < 0$ , and the proof of Theorem 1.1 is reduced to showing that the index of  $G_b - z$  is equal to zero. In Section 3 we collect some facts about the scattering operator  $S$  and the scattering matrix  $\mathcal{S}(z)$ . By applying a representation of the scattering kernel, we prove that  $\mathcal{S}(z)$  is invertible in a small neighborhood of 0. In Section 4 we prove that  $G_b$  has no outgoing eigenfunctions and  $G_b^*$  has no incoming eigenfunctions. This is a corollary of Theorem 4.2 which has independent interest since one obtains some conditions for the existence of disappearing solutions. Next we apply the characterization of the point spectrum of  $G_b$  in  $\operatorname{Re} z < 0$  given in [10], and we prove Theorem 1.1 by using the invertibility of  $\mathcal{S}(0)$ . In Section 5 we establish a result showing that the (ADS) are stable under the perturbations of the operator, the boundary and the boundary conditions.

**Acknowledgment.** We thank Jean-François Bony who showed us the construction of the symbol  $Q(\xi)$  in Section 4.

## 2. THE FREDHOLM OPERATOR $G_b - z$

Define  $\mathcal{H}_0 := (\operatorname{Ker} G_b)^\perp$ . For  $\operatorname{Re} z < 0$ , it is clear that  $G_b - z : \mathcal{H}_0 \cap D(G_b) \longrightarrow \mathcal{H}_0$ .

**Proposition 2.1.** *For every  $R > \rho$  there is a constant  $C_R > 0$  so that for every  $u \in \mathcal{H}_0 \cap D(G_b)$  and  $\operatorname{Re} z < 0$ ,*

$$(2.1) \quad \|u\|_{H^1(\Omega)} \leq C_R \left( |z| \left( 1 + \frac{1}{|\operatorname{Re} z|} \right) \right) (\|(G_b - z)u\| + \|\mathbf{1}_{|x| \leq 2R} u\|).$$

*Proof.* The coercive estimate yields

$$\|u\|_{H^1} \leq C(\|G_b u\| + \|u\|) \leq C(1 + |z|)(\|G_b - z\|u\| + \|u\mathbf{1}_{|x| < 2R}\| + \|u\mathbf{1}_{|x| \geq 2R}\|).$$

Estimate  $\|\mathbf{1}_{|x| \geq 2R}u\|$  as follows. Choose  $\varphi \in C^\infty(\mathbb{R}^n)$  so that  $\varphi = 0$  for  $|x| \leq R$ , and  $\varphi = 1$  for  $|x| \geq 2R$ . Define  $f := (G_b - z)u$ . Then  $G_b = G$  on the support of  $\varphi$  and

$$(G - z)(\varphi u) = \varphi f + [G, \varphi]u.$$

Since  $\|(G - z)^{-1}\| \leq \frac{1}{|\operatorname{Re} z|}$ ,

$$\|\varphi u\| \leq \frac{1}{|\operatorname{Re} z|} (\|\varphi f\| + \|[G, \varphi]u\|) \leq \frac{C_1}{|\operatorname{Re} z|} (\|f\| + \|\mathbf{1}_{|x| \leq 2R}u\|).$$

On the other hand,  $\mathbf{1}_{|x| \geq 2R}u = \mathbf{1}_{|x| \geq 2R}\varphi u$ , and this yields the estimate.  $\square$

**Corollary 2.2.** *For  $\operatorname{Re} z < 0$ ,  $\dim \operatorname{Ker}(G_b - z) < \infty$  and the range of the operator  $(G_b - z) : \mathcal{H}_0 \cap D(G_b) \rightarrow \mathcal{H}_0$  is closed.*

*Proof.* Suppose  $\operatorname{Re} z < 0$  and define  $B_0 := \{x \in \operatorname{Ker}(G_b - z) : \|x\| = 1\}$ . Then  $B_0 \subset \mathcal{H}_0$ . Proposition 2.1 implies that  $B_0$  is compact, hence  $\operatorname{Ker}(G_b - z)$  is finite dimensional.

Write  $\mathcal{H}_0 = \operatorname{Ker}(G_b - z) \oplus_\perp B_1$ . Then with a constant  $c > 0$ ,

$$(2.2) \quad \|u\| \leq c\|(G_b - z)u\|, \quad u \in B_1 \cap D(G_b).$$

If this inequality were not true, there would exist a sequence  $v_j \in B_1 \cap D(G_b)$ ,  $\|v_j\| = 1$  with  $(G_b - z)v_j \rightarrow 0$ .

Applying (2.1), it follows that there exists a subsequence  $v_{j_k}$  converging to  $w \in B_1$  with  $\|w\| = 1$ . Since  $(G_b - z)$  is a closed operator,  $(G_b - z)w = 0$ . Therefore  $w \in \operatorname{Ker}(G_b - z) \cap B_1 = \{0\}$ , so  $w = 0$ . This is a contradiction.

To prove that the range is closed, suppose that  $(G_b - z)u_j \rightarrow g$  with  $u_j = w_j + v_j$ ,  $w_j \in \operatorname{Ker}(G_b - z)$ ,  $v_j \in B_1 \cap D(G_b)$ . Obviously,  $(G_b - z)v_j \rightarrow g$  and by (2.2) the sequence  $\|v_j\|$  is bounded. Then by (2.1) there exists a convergent subsequence  $v_{j_k} \rightarrow y$ . Since  $(G_b - z)$  is a closed operator, this implies  $(G_b - z)y = g$ . Thus the range of  $(G_b - z)$  is closed.  $\square$

The same argument works for  $\dim \operatorname{Ker}(G_b^* - z)$  and  $\operatorname{Range}(G_b^* - z)$ . In particular, the codimension of  $\operatorname{Range}(G_b - z)$  is finite. Consequently, if  $z \in \sigma(G_b)$ ,  $\operatorname{Re} z < 0$  and  $z$  is not an eigenvalue of  $G_b$ , it follows that  $\operatorname{Range}(G_b - z) \neq \mathcal{H}$ . This means that  $z$  is in the residual spectrum of  $G_b$  and therefore  $\bar{z}$  is an eigenvalue of  $G_b^*$ .

Recall the definition of Fredholm operators.

**Definition 2.3.** A bounded operator  $L : X \rightarrow Y$  from a Banach space  $X$  to  $Y$  is called Fredholm if its kernel is finite dimensional and its range is closed and of finite codimension. The index of a Fredholm operator  $L$  is defined as the difference  $\operatorname{codim} \operatorname{Range}(L) - \dim \operatorname{Ker}(L)$ .

In the following  $\mathcal{H}_j$  denote Hilbert spaces and  $\mathcal{O}$  a connected open subset of  $\mathbb{C}$ . The next analytic Fredholm theorem is proved for the sake of completeness.

**Theorem 2.4.** *Suppose that  $L(\tau) : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is an analytic function of  $\tau \in \mathcal{O}$  with values in the space of bounded operators from  $\mathcal{H}_1$  to  $\mathcal{H}_2$ . Suppose that for all  $\tau \in \mathcal{O}$ , the operator  $L(\tau)$  is Fredholm. Then the set*

$$\{\tau \in \mathcal{O} : L(\tau) \text{ is invertible}\}$$

*is either empty or an open subset with discrete complement. In the latter case the operator valued function  $L(\tau)^{-1}$  is meromorphic on  $\mathcal{O}$ .*

*Proof.* The index of  $L(\tau)$  is independent of  $\tau$ . If the index is not zero, then  $A(\tau)$  is never invertible, so it suffices to treat the case of  $L(\tau)$  of index zero.

For any  $\underline{\tau} \in \mathcal{O}$ , it suffices to show that there is an  $r > 0$  so that the disk  $\mathbb{D}$  of radius  $r$  centered at  $\underline{\tau}$  is contained in  $\mathcal{O}$  and that either  $L(\tau)$  is nowhere invertible on  $\mathbb{D}$  or  $L(\tau)$  is invertible on the complement of a discrete subset in  $\mathbb{D}$  with  $L^{-1}(\tau)$  meromorphic. The proof reduces to the case of finite dimensions, where the result is elementary.

If  $L(\underline{\tau})$  is invertible, then the second alternative holds with the singular set being empty. So it suffices to consider the case when  $L(\underline{\tau})$  is singular.

Decompose

$$\mathcal{H}_1 = \ker L(\underline{\tau}) \oplus (\ker L(\underline{\tau}))^\perp, \quad \mathcal{H}_2 = (\operatorname{rg} L(\underline{\tau}))^\perp \oplus \operatorname{rg} L(\underline{\tau}).$$

Since  $L(\underline{\tau})$  is Fredholm of index zero, the first two summands are finite dimensional and of equal dimensions. Corresponding to these decompositions in the initial and final spaces, the operator  $L(\tau)$  has matrix

$$\begin{pmatrix} a_{11}(\tau) & a_{12}(\tau) \\ a_{21}(\tau) & a_{22}(\tau) \end{pmatrix}$$

with

$$a_{11}(\underline{\tau}) = 0, \quad a_{21}(\underline{\tau}) = 0, \quad a_{22}(\underline{\tau}) \text{ invertible}.$$

$L(\tau)$  is not invertible if and only if there is a nontrivial solution  $u = (u_1, u_2) \in \mathcal{H}_1$  to  $L(\tau)(u_1, u_2) = 0$ . The equation  $L(\tau)(u_1, u_2) = 0$  is equivalent to the pair of identities,

$$a_{11}u_1 + a_{12}u_2 = 0, \quad a_{21}u_1 + a_{22}u_2 = 0.$$

For  $0 < r$  sufficiently small,  $\mathbb{D} := \{|\tau - \underline{\tau}| < r\} \subset \mathcal{O}$  and  $a_{22}(\tau)$  is invertible with analytic inverse. For those values of  $\tau$  the second equation holds if and only if  $u_2 = -a_{22}^{-1}a_{21}u_1$ . Inserting this in the first equation, shows that the pair has a nontrivial solution if and only if there is a nontrivial solution  $u_1$  of the equation

$$(a_{11} - a_{12}a_{22}^{-1}a_{21})u_1 = 0.$$

The finite dimensional theorem implies that either  $a_{11} - a_{12}a_{22}^{-1}a_{21}$  is nowhere invertible on  $\mathbb{D}$  or it is invertible on the complement of a discrete set with meromorphic inverse. This suffices to complete the proof.  $\square$

**Example 2.5.** Take  $\mathcal{H}_1$  to be the set  $(H^1(\Omega))^r$  of vector valued functions satisfying the dissipative boundary condition  $f|_{\partial\Omega} \in \mathcal{N}(x)$  and belonging to  $\mathcal{H}_0$ . Let  $\mathcal{H}_2$  be the set of vector valued  $(L^2(\Omega))^r$  functions. When hypotheses  $(H)$  and  $(H)^*$  are satisfied,  $L(\tau) := G_b - \tau : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is an analytic family of bounded operators and the graph norm of  $G_b$  is equivalent to the  $H^1(\Omega)$  norm. The fact that  $G_b - \tau :$

$\mathcal{H}_1 \longrightarrow \mathcal{H}_2$  is Fredholm follows from Corollary 2.2. In Section 4 we will prove that the resolvent set of  $G_b$  in  $\mathcal{O} := \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$  is not empty. This implies that  $G_b - \tau$  is invertible on the complement of a discrete subset of finite multiplicity eigenvalues in  $\mathcal{O}$ .

*Remark 2.6.* The Fredholm property fails on the imaginary axis, as it does for the unperturbed operator.

### 3. SCATTERING MATRIX $\mathcal{S}(z)$ , INVERTIBILITY OF $\mathcal{S}(0)$

Introduce the wave operators related to  $U_0(t)$  and  $V(t)$ , assuming that  $(H)$  and  $(H)^*$  are satisfied. Consider the operator  $J : \mathcal{H} \longrightarrow H_0$  extending  $f \in \mathcal{H}$  as 0 for  $x \in K$  and let  $J^* : H_0 \longrightarrow \mathcal{H}$  be the adjoint of  $J$ . Let  $H_0^{ac} = (\operatorname{Ker} G_0)^\perp$ . The wave operators are defined by

$$W_- f = \lim_{t \rightarrow +\infty} V(t) J^* U_0(-t) f \quad W_+ f = \lim_{t \rightarrow +\infty} V^*(t) J^* U_0(t) f, \quad f \in H_0^{ac}.$$

It is not difficult to prove the existence of  $W_\pm$  (see, for instance, [5] and Chapter III in [13]). In addition,

$$\overline{\operatorname{Ran} W_\pm} \subset \mathcal{H}_b^\perp.$$

It was proved in [12] that if  $G_b$  has eigenvalue in  $\operatorname{Re} z < 0$  with eigenfunction  $f$  whose support is not compact, then the wave operators are not complete and  $\operatorname{Ran} W_- \neq \operatorname{Ran} W_+$ . In the next section we prove that the eigenfunctions with eigenvalues in  $\operatorname{Re} z < 0$  never have compact support.

The corresponding operators with the roles of free and perturbed interchanged are

$$Wf := \lim_{t \rightarrow \infty} U_0(-t) J V(t) f, \quad W_1 f := \lim_{t \rightarrow \infty} U_0(t) J V^*(t) f, \quad f \in \mathcal{H}_b^\perp.$$

In [5], [13], the existence of these limits is proved assuming the hypothesis  $(H)$ ,  $(H)^*$ .

The Hilbert spaces  $H_0^{ac}$  and  $\mathcal{H}_b^\perp$  are invariant under the action of  $U_0(t)$  and  $V(t)$  respectively. This is the setting of two spaces scattering theory with scattering operator defined as  $S := W \circ W_-$ . We have  $\overline{\operatorname{Ran} W} \subset H_0^{ac}$  and  $S$  is a bounded operator from  $H_0^{ac}$  to  $H_0^{ac}$ .

We recall some elements of Lax-Phillips scattering theory that are needed. The translation representation

$$\mathcal{R}_n : H_0^{ac} \longrightarrow (L^2(\mathbb{R} \times \mathbb{S}^{n-1}))^d$$

of  $U_0(t)$ , involves both the Radon transform  $(Rf)(s, \omega)$  of  $f(x)$  and eigenvalue-eigenvector pairs  $\tau_j(\xi), r_j(\xi)$ . With  $\operatorname{Rank} A(\xi) = r - d_0 = 2d > 0$  for  $\xi \neq 0$ , define  $\tau_j(\xi)$ ,  $j = 1, \dots, 2d$ , to be the non-vanishing eigenvalues of  $A(-\xi)$  in decreasing order,

$$\tau_1(\xi) \geq \dots \geq \tau_d(\xi) > 0 > \tau_{d+1}(\xi) \geq \dots \geq \tau_{2d}(\xi), \quad \xi \neq 0.$$

Choose measurable normalized eigenvectors  $r_j(\xi)$ ,  $j = 1, \dots, 2d$ , of  $A(-\xi)$  with eigenvalues  $\tau_j(\xi)$ . Then  $\mathcal{R}_n$  has the form (see Chapter VI in [9] for  $d_0 = 0$  and Chapter

IV in [13] for  $d_0 > 0$ )

$$(\mathcal{R}_n f)(s, \omega) = \sum_{j=1}^d \tilde{k}_j(s, \omega) r_j(\omega), \quad \tilde{k}_j(s, \omega) := \tau_j(\omega)^{1/2} k_j(s \tau_j(\omega), \omega), \quad j = 1, \dots, d,$$

where

$$k_j(s, \omega) := 2^{-(n-1)/2} D_s^{(n-1)/2} \langle (Rf)(s, \omega), r_j(\omega) \rangle.$$

$\mathcal{R}_n$  is an isometry  $H_0^{ac} \rightarrow (L^2(\mathbb{R} \times \mathbb{S}^{n-1}))^d$ , and  $\mathcal{R}_n U_0(t) = T_t \mathcal{R}_n$ ,  $\forall t \in \mathbb{R}$ , where  $T_t g = g(s - t, \omega)$ .

**Definition 3.1.** Define the slowest nonzero sound speed  $v_{min} := \min_{|\omega|=1} \tau_d(\omega) > 0$ . The set  $D_\pm \subset H_0$  is the set of  $f$  satisfying

$$U_0(t)f = 0 \quad \text{for } |x| < \pm v_{min} t, \quad \pm t > 0.$$

Equivalently  $f \in D_\pm$  if and only if  $\mathcal{R}_n f(s, \omega) = 0$  for  $\mp s > 0$ . Define

$$D_\pm^a := U_0(\pm a/v_{min}) D_\pm, \quad a \geq \rho.$$

Then  $D_\pm^\rho \subset \mathcal{H}_b^\perp$  (see Lemma 4.1.6 in [13]),

$$J(\overline{D_+^\rho \oplus D_-^\rho}) = \overline{D_+^\rho \oplus D_-^\rho},$$

$$J^*(\overline{D_+^\rho \oplus D_-^\rho}) = \overline{D_+^\rho \oplus D_-^\rho}.$$

It follows that the operators  $W_\pm$  and  $W$  defined above coincide with the Lax-Phillips wave operators [10] related to  $H_0^{ac}$  and  $\mathcal{H}_b^\perp$ .

Consider  $D_\pm^a$ ,  $a \geq \rho$ , as subspaces of  $\mathcal{H}_b^\perp$  and introduce the orthogonal projectors  $P_\pm^a$  on the orthogonal complements of  $D_\pm^a$  in  $\mathcal{H}_b^\perp$ . It is easy to see (cf. Chapter IV, [13]) that

$$(i) \quad V(t)D_+^\rho \subset D_+^\rho \quad \text{and} \quad V^*(t)D_-^\rho \subset D_-^\rho, \quad t \geq 0,$$

$$(ii) \quad \bigcap_{t \geq 0} V(t)D_+^\rho = \{0\} \quad \text{and} \quad \bigcap_{t \geq 0} V^*(t)D_-^\rho = \{0\}.$$

Next we prove

$$(iii) \quad \lim_{t \rightarrow \infty} P_+^\rho V(t)f = 0 \quad \forall f \in \mathcal{H}_b^\perp \quad \text{and} \quad \lim_{t \rightarrow \infty} P_-^\rho V^*(t)f = 0, \quad \forall f \in \mathcal{H}_b^\perp.$$

The existence of the operator  $W$  implies the first assertion. In fact, given  $g \in \mathcal{H}_b^\perp$ , there exists  $f \in H_0^{ac}$  such that

$$\lim_{t \rightarrow \infty} \|V(t)g - (U_0(t)f)|_{\mathcal{H}_b^\perp}\|_H = 0.$$

Thus it suffices to prove that  $\lim_{t \rightarrow \infty} \|P_+^\rho(U_0(t)f)|_{\mathcal{H}_b^\perp}\|_H = 0$ . It is clear that this holds for  $f = U_0(s)h$ ,  $h \in D_+^\rho$ . Since

$$\bigcup_{t \in \mathbb{R}} U_0(t)D_+^\rho = H_0^{ac},$$

the assertion follows for every  $f \in H_0^{ac}$ . Thus  $\lim_{t \rightarrow \infty} P_+^\rho V(t)g = 0$ .

For the other relation in (iii), use the existence of the operator  $W_1$ .

Properties (i)-(iii), justify the application of the abstract Lax-Phillips scattering theory for dissipative operators [10].



Using the translation representation  $\mathcal{R}_n$  of  $U_0(t)$ , consider the scattering operator

$$\tilde{S} : (L^2(\mathbb{R} \times \mathbb{S}^{n-1}))^d \rightarrow (L^2(\mathbb{R} \times \mathbb{S}^{n-1}))^d, \quad \tilde{S} := \mathcal{R}_n S \mathcal{R}_n^{-1}.$$

The kernel of  $\tilde{S} - I$  is a matrix-valued distribution  $K^\#(s - s', \theta, \omega)$  and

$$K^\#(s, \theta, \omega) = \left( S^{ji}(s, \theta, \omega) \right)_{j,i=1}^d$$

is called *scattering kernel*. The relation  $(S - I)(D_-^\rho) \subset (D_+^\rho)^\perp$  (see Lemma 4.1.7 in [13]) implies

$$K^\#(s, \theta, \omega) = 0 \quad \text{for } s > 2\rho/v_{\min},$$

and the Fourier transform  $\hat{K}^\#(z, \theta, \omega)$  of  $K^\#(s, \theta, \omega)$  with respect to  $s$  is analytic for  $\text{Im } z < 0$ . The same is true for the operator-valued function  $\mathcal{S}(z) - I$  with kernel  $\hat{K}^\#(z, \theta, \omega)$ . The operator  $\mathcal{S}(z)$  is called the *scattering matrix* (cf. for instance [10]). More precise information about the support of  $K^\#(s, \theta, \omega)$  and the maximal singularity of  $K^\#(s, -\omega, \omega)$  is given in [11], [12] for the case  $d_0 = 0$  and in [14] for the case studied in this paper. In particular, in [14] one proves a representation of the scattering kernel and taking the Fourier transform with respect to  $s$  it follows that  $\mathcal{S}(z) = I + \mathcal{K}(z)$  for  $\text{Im } z < 0$  is an analytic operator-valued function and  $\mathcal{K}(z)$  is an operator-valued function with values in the space of Hilbert Schmidt operators in  $(L^2(\mathbb{S}^{n-1}))^d$ .

To obtain a representation of  $\mathcal{K}(z)$ , recall Theorem 15.5 in [14]. This result was proved when  $A(\xi)$  has characteristic roots with constant multiplicities for  $\xi \neq 0$ . The proof works without any changes when only zero is a characteristic root of constant multiplicity. Let

$$w_k^o(t, x, \omega) = \tau_k(\omega)^{1/2} \delta^{(n-1)/2}(\langle x, \omega \rangle - \tau_k(\omega)t) r_k(\omega), \quad k = 1, \dots, d,$$

and consider the solution  $w_k^s(t, x, \omega)$  of the problem

$$(3.1) \quad \begin{cases} (\partial_t - G)w_k^s = 0 \text{ in } \mathbb{R} \times \Omega, \\ w_k^s + w_k^o \in \mathcal{N}(x) \text{ on } \mathbb{R} \times \partial\Omega, \\ w_k^s|_{t \leq -\frac{\rho}{v_{\min}}} = 0. \end{cases}$$

The scattering kernel  $K^\#(s, \theta, \omega)$  computed with respect to the basis  $\{r_j(\omega)\}_{j=1}^d$  has matrix elements  $S^{jk}(s, \theta, \omega)$  equal to

$$(3.2) \quad d_n^2 \tau_j(\theta)^{1/2} \int_{\mathbb{R} \times \partial\Omega} \delta^{(n-1)/2}(\langle x, \theta \rangle - \tau_j(\theta)(s+t)) \left\langle r_j(\theta), A(\nu(x))(w_k^o + w_k^s)(t, x, \omega) \right\rangle dt dS_x,$$

where  $d_n \neq 0$  is a constant depending only on  $n$  and the integral is taken in sense of the distributions. Thus for  $\text{Im } z < 0$ , taking the Fourier transform with respect to  $s$ , one obtains

$$(3.3) \quad \begin{aligned} \hat{S}^{ji}(z, \theta, \omega) &= (-iz)^{(n-1)/2} d_n^2 \tau_j(\theta)^{1/2} \\ &\times \int_{\mathbb{R} \times \partial\Omega} \exp\left(iz \left(\frac{\langle x, \theta \rangle}{\tau_j(\theta)} - t\right)\right) \left\langle r_j(\theta), A(\nu(x))(w_k^o + w_k^s)(t, x, \omega) \right\rangle dt dS_x. \end{aligned}$$

The above equality for  $\text{Im } z < 0$  yields

$$(3.4) \quad \hat{K}^\#(z, \theta, \omega) = c_n z^{(n-1)/2} K_1(z, \theta, \omega), \quad c_n \neq 0$$

with  $K_1(z, \theta, \omega)$  an analytic matrix-valued function for  $\text{Im } z < 0$ .

Since  $w_k^o(t, x, \omega)|_{\mathbb{R} \times \partial\Omega}$  has compact support with respect to  $t$ , the integral

$$\int_{\mathbb{R} \times \partial\Omega} \exp\left(\mathbf{i}z\left(\frac{\langle x, \theta \rangle}{\tau_j(\theta)} - t\right)\right) \left\langle r_j(\theta), A(\nu(x))w_k^o(t, x, \omega) \right\rangle dt dS_x,$$

yields an analytic function in  $\mathbb{C}$ . In the integral in (3.3) over  $\mathbb{R}$  involving  $w_k^s$ , for  $\text{Im } z < 0$  take the inverse Fourier transform  $\mathcal{F}_{t \rightarrow z}$  with respect to  $t$  in the sense of distributions and denote

$$v_k^s(z, x, \omega) = \mathcal{F}_{t \rightarrow z} w_k^s(t, x, \omega).$$

Then  $v_k^s(z, x, \omega)$  is a solution of the problem

$$\begin{cases} (\mathbf{i}z - G)v_k^s = 0 \text{ in } \Omega, \\ \left( v_k^s + (\mathbf{i}z)^{(n-1)/2} e^{-\mathbf{i}z \frac{\langle x, \omega \rangle}{\tau_k(\omega)}} r_k(\omega) \right) \Big|_{\partial\Omega} \in \mathcal{N}(x) \text{ on } \partial\Omega. \end{cases}$$

Let  $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$  be a function such that  $\varphi(x) = 1$  for  $|x| \leq \rho$ ,  $\varphi(x) = 0$  for  $|x| \geq 2\rho$ . Write  $v_k^s(z, x, \omega) = V_k^s(z, x, \omega) - (\mathbf{i}z)^{(n-1)/2} \varphi(x) e^{-\mathbf{i}z \frac{\langle x, \omega \rangle}{\tau_k(\omega)}} r_k(\omega)$  and deduce

$$\begin{cases} (\mathbf{i}z - G)V_k^s = -[G, \varphi](\mathbf{i}z)^{(n-1)/2} e^{-\mathbf{i}z \frac{\langle x, \omega \rangle}{\tau_k(\omega)}} r_k(\omega) \text{ in } \Omega, \\ V_k^s \in \mathcal{N}(x) \text{ on } \partial\Omega. \end{cases}$$

Consequently, setting  $-[G, \varphi] = \psi(x)$ , one has

$$V_k^s(z, x, \omega) = (\mathbf{i}z - G_b)^{-1} \psi(x) (\mathbf{i}z)^{(n-1)/2} e^{-\mathbf{i}z \frac{\langle x, \omega \rangle}{\tau_k(\omega)}} r_k(\omega)$$

and

$$V_k^s(z, x, \omega) \Big|_{\partial\Omega} = (\mathbf{i}z)^{(n-1)/2} \left[ \varphi(x) (\mathbf{i}z - G_b)^{-1} \psi(x) e^{-\mathbf{i}z \frac{\langle x, \omega \rangle}{\tau_k(\omega)}} r_k(\omega) \right] \Big|_{\partial\Omega}.$$

In conclusion, the kernel  $K_1(z, \theta, \omega)$  for  $\text{Im } z < 0$  is given by a sum of a function analytic with respect to  $z$  in  $\mathbb{C}$  and an integral over  $\partial\Omega$  involving the cut-off resolvent  $\varphi(x)(\mathbf{i}z - G_b)^{-1} \psi(x)$ . A similar argument has been used for the scattering amplitude  $s(z, \theta, \omega)$  related to the wave equation in [15].

To introduce the scattering resonances, consider in  $\mathcal{H}_b^\perp$  the semigroup of contractions  $Z(t)$  in the space  $K^a := \mathcal{H}_b^\perp \ominus (D_-^a \oplus D_+^a)$ ,  $a \geq \rho$ ,

$$Z_a(t) := P_+^a V(t) P_-^a := e^{tB_a}, \quad t \geq 0.$$

Exploiting the condition (H), as in [9], [10], it follows that the spectrum of  $B_a$  is discrete and formed only by isolated eigenvalues of finite multiplicity in  $\text{Re } z < 0$  and this spectrum is independent on the choice of  $a$ . Corollary 4.8 in [10] implies that *the resolvent of  $B_a$  is a meromorphic function in  $\mathbb{C}$* . Now choose  $a \geq 2\rho$ . Then  $\varphi P_+^a = \varphi$ ,  $P_-^a \psi(x) = \psi(x)$  and for  $\text{Im } z < 0$ ,

$$\varphi(\mathbf{i}z - B_a)^{-1} \psi = \int_0^\infty \varphi P_+^a V(t) P_-^a \psi dt = \varphi(\mathbf{i}z - G_b)^{-1} \psi.$$

The left hand side is analytic for  $\text{Im } z \leq 0$ , so  $\varphi(\mathbf{i}z - G_b)^{-1}\psi$  is analytic for  $\text{Im } z \leq 0$ . This implies that  $\lim_{z \rightarrow 0, \text{Im } z < 0} K_1(z, \theta, \omega) = K_1(0, \theta, \omega)$  is a kernel of a bounded operator in  $(L^2(\mathbb{S}^{n-1}))^d$  and  $\mathcal{S}(0)$  is invertible.

#### 4. THE SPECTRUM OF $G_b$

**Fix**  $a = 2\rho$  and write simply  $Z(t)$ ,  $B$  instead of  $Z_a(t)$ ,  $B_a$ .

Passing to the translation representation, yields the decomposition  $\mathcal{H}_b^\perp = K^a \oplus D_-^a \oplus D_+^a$ .

**Definition 4.1.** A function  $f$  is **outgoing** (resp. **incoming**) if the component of  $f$  in  $D_-^a$  (resp.  $D_+^a$ ) vanishes.

In [10] the incoming (outgoing) functions are defined with respect to the components in  $D_+^\rho$  ( $D_-^\rho$ ). For our analysis the role of  $\rho$  is played by  $a = 2\rho$ . To apply the results of Section 5 in [10], we need to prove that neither  $G_b$  nor  $G_b^*$  have eigenfunctions in  $\mathcal{H}_b^\perp$  that are both incoming and outgoing. A stronger result is Corollary 4.5 below. First we prove the following

**Theorem 4.2.** *Let  $f \in H_b^\perp$ . Then the following conditions are equivalent.*

- (a) *there exists  $b \geq 2\rho$  such that  $f \perp D_-^b$  and  $\lim_{t \rightarrow \infty} V(t)f = 0$ .*
- (b)  *$V(t)f$  is a disappearing solution.*

This results is similar to Theorem 1 of Georgiev [4] (see also Theorem 4.3.2 in [13]) established in the case  $\det A(\xi) \neq 0$  for  $\xi \neq 0$ . For non elliptic symmetric systems new ideas are needed.

*Proof of Theorem 4.2.* The implication (b)  $\Rightarrow$  (a) follows from the finite speed of propagation. We prove that (a)  $\Rightarrow$  (b). Assume (a). First treat the case when  $f \in \cap_{j=1}^\infty (G_b)^j \cap \mathcal{H}_b^\perp \cap (D_-^b)^\perp$ . Set  $u(t, x) = V(t)f$  and consider the solution

$$\tilde{u}(t, x) := V(t)G_b f = G_b V(t)f = \partial_t(V(t)f).$$

Since  $f \perp D_-^\rho$ , we have  $V(t)f \perp D_-^\rho$ ,  $t \geq 0$ . Taking the derivative with respect to  $t$ , yields  $\tilde{u}(t, x) \perp D_-^\rho$ .

We claim that  $\lim_{t \rightarrow +\infty} \tilde{u}(t, x) = 0$ . Our hypothesis implies

$$\|\partial_t^j(V(t)f)\| = \|V(t)(G_b)^j f\| \leq C_j, \quad \forall t \geq 0, \forall j \in \mathbb{N}.$$

To prove the claim, suppose that there exists a sequence  $t_k \rightarrow +\infty$  such that  $\|u_t(t_k, x)\| \geq \delta > 0$ ,  $\forall k \in \mathbb{N}$ . Then for  $\xi \leq \eta \leq t_k$  we get

$$\|u_t(t_k, x) - u_t(\eta, x)\| = \left\| \int_\eta^{t_k} u_{tt}(y, x) dy \right\| \leq C_2(t_k - \xi)$$

and, provided  $0 \leq t_k - \xi \leq \frac{\delta}{2C_2}$ , one has

$$\|u(t_k, x) - u(\xi, x)\| = \left\| \int_\xi^{t_k} u_t(y, x) dy \right\| \geq (t_k - \xi)\delta - (t_k - \xi)^2 C_2 \geq (t_k - \xi) \frac{\delta}{2}.$$

This contradicts the hypothesis  $u(t_k, x) \rightarrow 0$  and the claim is proved.

Choose a function  $\varphi(x) \in C^\infty(\mathbb{R}^n)$  with  $\varphi(x) = 1$  for  $|x| \geq 2\rho$ ,  $\varphi(x) = 0$  for  $|x| \leq \rho$ . With  $v(t, x) := \varphi(x)\tilde{u}(t, x)$  compute

$$v(t, x) = \varphi V(t)G_b f = G\varphi V(t)f + [\varphi, G]V(t)f := w(t, x) + [\varphi, G]V(t)f,$$

the last equality defining  $w$ .

Next we will prove that  $w(t, x) = 0$  for  $|x| \geq b$ ,  $t \geq 0$ . Start with  $G_b\varphi = G\varphi$ . It is clear that  $w(t, x) \in (\text{Ker } G)^\perp = H_0^{ac}$ , so we may consider the translation representation  $\mathcal{R}_n(w(t, x)) = m(t, s, \omega)$  of  $w(t, x)$ . Since  $f \in H_b^\perp$ , there exists a sequence  $t_k \rightarrow \infty$  such that  $\lim_{t_k \rightarrow \infty} \|V(t_k)f\|_{L^2(|x| \leq 2\rho)} = 0$  (see Proposition 3.1.9 in [13]). On the other hand,  $\|\tilde{u}(t_k, x)\|_{L^2(\Omega)} \rightarrow 0$  as  $t_k \rightarrow \infty$ , so  $\lim_{t_k \rightarrow \infty} \|w(t_k, x)\|_{L^2(\Omega)} = 0$ .

Write, with the last equality defining  $g$ ,

$$(\partial_t - G)w(t, x) = - \sum_{j=1}^n (A_j \varphi_{x_j}) V(t) G_b f + [G, [\varphi, G]] V(t) f := g(t, x).$$

Then  $g(t, x) \in (L^2(\mathbb{R}^+ \times \mathbb{R}^n))^r$  vanishes for  $|x| \geq 2\rho$ . Applying the transformation  $\mathcal{R}_n$  to both sides of the above equality and setting

$$m_j(t, s, \omega) = \langle m(t, s, \omega), r_j(\omega) \rangle, \quad l_j(t, s, \omega) = \langle \mathcal{R}_n(g)(t, s, \omega), r_j(\omega) \rangle, \quad j = 1, \dots, d,$$

yields

$$(\partial_t + \tau_j(\omega)\partial_s)m_j(t, s, \omega) = l_j(t, s, \omega), \quad j = 1, \dots, d.$$

Fix  $t_1 \geq 0$  and  $1 \leq j \leq d$ . Since  $l_j(t, s, \omega) = 0$  for  $|x| \geq 2\rho$ ,  $t \geq t_1$  and  $s \geq 2\rho + \tau_j(\omega)(t - t_1)$ , the above equations yield

$$m_j(t, s, \omega) = m_j(t_1, s - \tau_j(\omega)(t - t_1), \omega).$$

This implies

$$\int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}} |m_j(t, s, \omega)|^2 ds d\omega \geq \int_{2\rho}^\infty \int_{\mathbb{S}^{n-1}} |m_j(t_1, s, \omega)|^2 ds d\omega.$$

Letting  $t_k \rightarrow \infty$ , one gets  $m_j(t, s, \omega) = 0$  for  $s \geq 2\rho$ . Since  $V(t)f \perp D_-^b$  and  $\text{supp}(1 - \varphi(x)) \subset \{|x| \leq 2\rho\}$ , this implies that  $w(t, x) \perp D_-^b$  for  $t \geq 0$ . Therefore,  $m(t, s, \omega) = \mathcal{R}_n(w(t, x)) = 0$  for  $s \leq -b$ .

Next repeat the argument of Lemma 2.2 in [4] based on the following

**Lemma 4.3** ([3]). *Let  $F \in \bigcap_{j=1}^\infty D(G^j) \cap H_0^{ac}$  and let  $\mathcal{R}_n F = 0$  for  $|s| \geq b$ . Then  $F = 0$  for  $|x| \geq b$  if and only if*

$$\int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}} [A(\omega)]^k [(\mathcal{R}_n F)(s, \omega) + (-1)^{(n-1)/2} (\mathcal{R}_n F)(-s, -\omega)] s^a Y_m(\omega) ds d\omega = 0$$

for  $a = 0, 1, 2, \dots$  and any spherical harmonic function  $Y_m(\omega)$  of order  $m \geq a + k + (3 - n)/2$ .

We conclude that  $w(t, x)$  and  $v(t, x)$  have support in  $\{x : |x| \leq b\}$  for all  $t \geq 0$ . Thus  $\tilde{u}(t, x) = \partial_t(V(t)f) = 0$  for  $|x| \geq b$ . Consequently, our hypothesis implies that  $u(t, x) = 0$  for  $|x| \geq b$ .

The next step is to show that  $\text{supp } u(t, x) \subset \{x : |x| \geq b\}$  implies that  $u$  is a disappearing solution. We know that  $\partial_t \tilde{u} - G\tilde{u} = 0$ . By using the hypothesis that

$A(\xi)$  has constant rank for  $\xi \neq 0$ , it is possible to show that there is an  $(r \times r)$  matrix-valued polynomial  $Q(\xi)$  such that

$$(4.1) \quad \text{Ker } Q(\xi) = \text{Ran } A(\xi), \quad \xi \neq 0.$$

For completeness we present a proof of (4.1). Recall that  $r - d_0 = 2d > 0$ . For  $\xi \neq 0$  let

$$P(z, \xi) = R(z, \xi)z^{d_0} = \prod_{j=1}^{2d} (z + \tau_j(\xi))z^{d_0}$$

be the characteristic polynomial of  $A(\xi)$ , where  $-\tau_j(\xi)$  are the non-vanishing eigenvalues of  $A(\xi)$  repeated with their multiplicities. Define  $Q(\xi) = R(A(\xi), \xi)$ ,  $\xi \neq 0$ .

Next show that  $Q(\xi)$  is a polynomial  $\sum_{|\alpha| \leq 2d} B_\alpha \xi^\alpha$  with matrix coefficients  $B_\alpha$ . Since  $A^j(\xi)$  are polynomials with matrix coefficients, it is sufficient to prove that  $R(z, \xi) = \sum_{j=0}^{2d} c_j(\xi)z^j$  has coefficients  $c_j(\xi)$  that are polynomials. This follows from

$$\det(zI - A(\xi)) = \sum_{j=0}^{2d} c_j(\xi)z^{j+d_0}$$

upon comparing the coefficients of  $z^{j+d_0}$  in both sides. The Cayley-Hamilton theorem implies that

$$P(A(\xi), \xi) = Q(\xi)A^{d_0}(\xi) = 0.$$

Passing to a diagonal form of  $A(\xi)$ , shows that  $\text{Ran } A(\xi) = \text{Ran } A^{d_0}(\xi)$ . Thus  $\text{Ran } A(\xi) \subset \text{Ker } Q(\xi)$ ,  $\xi \neq 0$ . To establish the opposite inclusion, assume that  $h \in \text{Ker } Q(\xi)$  and write  $h = h_1 + h_2$  with  $h_1 \in \text{Ker } A(\xi)$ ,  $h_2 \in \text{Ran } A(\xi)$ . Then  $Q(\xi)h = \prod_{j=1}^{2d} \tau_j(\xi)h_1 = 0$  and we conclude that  $h_1 = 0$ .

Let  $Q(D_x)$  be the operator with symbol  $Q(\xi)$  and let  $L = (\partial_t - G)^{4d} + Q^2$ . Then  $Q\tilde{u} = QGV(t)f = 0$  and we get  $L\tilde{u} = 0$ . The symbol of the operator  $L$  with constant coefficients is

$$(\tau I - A(\xi))^{4d} + Q^2(\xi) = l(\tau, \xi).$$

First, show that  $\det l(0, \xi) \neq 0$  for  $\xi \neq 0$ . In fact, if for  $\xi_0 \neq 0$  there exists a vector  $v \neq 0$  such that  $A^{4d}(\xi_0)v + Q^2(\xi_0)v = 0$ , taking the scalar product by  $v$ , yields  $A(\xi_0)v = Q(\xi_0)v = 0$ . This implies  $v \in \text{Ker } A(\xi_0) \cap \text{Ran } A(\xi_0)$  and therefore  $v = 0$  which is a contradiction. Next, if for  $\tau \neq 0$  and  $\xi \neq 0$  we have  $\det l(\tau, \xi) = 0$ , by the same argument we deduce that  $\det(\tau I - A(\xi)) = 0$  and therefore  $\tau = -\tau_j(\xi)$  for some  $j = 1, \dots, 2d$ . If  $\tau < 0$ , we have  $\tau = -\tau_j(\xi)$  for some  $j = 1, \dots, d$ . On the other hand, if  $\tau > 0$ , we get  $\tau = -\tau_j(\xi) = \tau_{2d-j+1}(-\xi)$  for some  $j = d+1, \dots, 2d$ . Thus

$$\frac{|\tau|}{|\xi|} \geq \min\{\tau_j(\omega), \omega \in \mathbb{S}^{n-1}, j = 1, \dots, d\} = v_{\min} > 0.$$

Denote by  $S(x, r)$  the ball  $\{y \in \mathbb{R}^n : |y - x| \leq r\}$ .

**Proposition 4.4.** *Suppose that  $S(z, r_0) \subset \Omega$ ,  $|z - z_1| = \frac{1}{2}r_0 > 0$  with  $0 < r_1 < r_0$ . Assume that  $U \in \mathcal{D}'(\mathbb{R}_t \times \Omega)$  is a solution of  $LU = 0$  such that  $U(t, x) = 0$  for  $t \geq t_0 \geq 0$ ,  $x \in S(z, \frac{1}{2}r_0)$ . Then*

$$(4.2) \quad U(t, x) = 0 \text{ for } x \in S(z_1, \frac{1}{2}r_1), \quad t \geq t_0 + \frac{1}{2v_{\min}}r_0.$$

*Proof.* Let  $\Pi$  be a characteristic hyperplane for  $L$  with normal  $N = (\tau, \xi) \in \mathbb{R}^{n+1} \setminus \{0\}$ . Then  $\det l(\tau, \xi) = 0$  so  $|\tau| \geq v_{\min}|\xi|$ . Thus, if  $|\tau| < v_{\min}|\xi|$  the hyperplane  $\Pi$  is not characteristic. Since  $r_1 < r_0$ , a simple geometric argument shows that for  $s \geq t_0$  every characteristic hyperplane for  $L$  which intersects the convex set  $\{(t, x) : |x - z| \leq \frac{1}{2}r_0 + v_{\min}\frac{r_1}{r_0}(t - s), s \leq t \leq s + \frac{1}{2v_{\min}}r_0\}$  intersects also the convex set  $\{(t, x) : |x - z| \leq \frac{1}{2}r_0, s \leq t \leq s + \frac{1}{2v_{\min}}r_0\}$ . Thus we can apply F. John's global Holmgren theorem (see for instance, Chapter 1, Corollary 9 in [16] or Theorem 8.6.8 in [6]) to conclude that  $U(t, x) = 0$  for  $x \in S(z_1, \frac{1}{2}r_1)$ ,  $t \geq t_0 + \frac{1}{2v_{\min}}r_0$ . For convenience of the reader the non-characteristic deformations of the boundary  $|x| = \frac{1}{2}r_0, t \geq 0$  are sketched in Figure 1.  $\square$

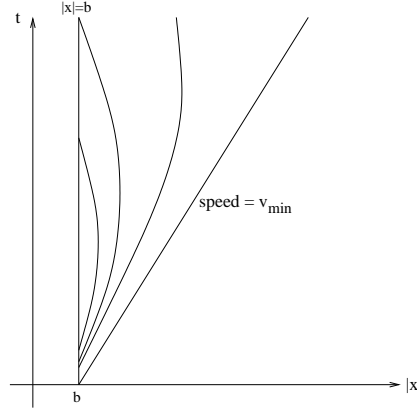


FIGURE 1. Deformation of the boundary

Now suppose that  $\tilde{u}(t, x) = 0$  for  $|x| \geq b, t \geq 0$ . Fix a point  $x_0$  and  $r_0$  so that  $S(x_0, \frac{1}{2}r_0) \subset \{|x| > b\}$  and  $S(x_0, r_0) \subset \Omega$ . Since  $\Omega$  is connected, given a point  $\hat{x} \in \Omega$ , there exists a path  $\Gamma_{\hat{x}} \subset \Omega$  with length less than a fixed number  $L_0 > 0$  independent on  $\hat{x}$  and points  $x_1, x_2, \dots, x_m = \hat{x}$  on the path  $\Gamma_{\hat{x}}$  so that

$$S(x_j, r_j) \subset \Omega, \quad j = 0, 1, \dots, m-1,$$

$$r_0 > r_1 > \dots > r_{m-1} > 0$$

with  $r_j = 2|x_{j+1} - x_j|$ ,  $j = 0, 1, \dots, m-1$ . Assume that

$$L\tilde{u} = 0 \quad \text{for } |x - x_j| \leq r_j, \quad t \geq 0,$$

$$\tilde{u}(t, x) = 0 \quad \text{for } x \in S(x_j, \frac{1}{2}r_j), \quad t \geq \frac{1}{2v_{\min}} \sum_{k=0}^{j-1} r_k.$$

Applying Proposition 4.4 with  $U = \tilde{u}$ , shows that

$$\tilde{u}(t, x) = 0 \quad \text{for } x \in S(x_{j+1}, \frac{1}{2}r_{j+1}), \quad t \geq \frac{1}{2v_{\min}} \sum_{k=0}^j r_k.$$

Consequently,  $\tilde{u}(t, \hat{x}) = 0$  for  $t \geq \frac{1}{2v_{\min}} \sum_{k=0}^{m-1} r_k = \frac{1}{2v_{\min}} L_0$ . This argument works for every  $x \in \Omega$ , hence  $\tilde{u}(t, x)$  as well  $u(t, x)$  are disappearing. Moreover, the constant  $T_0 = \frac{1}{2v_{\min}} L_0$  depends on  $b$  and  $\Omega$ , so it is independent on  $f$ .

Next treat the general case. For every fixed  $\epsilon > 0$ , we construct a sequence  $\{\varphi_\epsilon\}$  such that

$$\varphi_\epsilon \in \mathcal{H}_b^\perp \cap \left( \bigcap_{j=1}^\infty D(G_b^j) \right) \cap D_-^\rho, \quad \lim_{t \rightarrow \infty} V(t)\varphi_\epsilon = 0$$

and  $\|\varphi_\epsilon - f\| < \epsilon$ . The construction of this sequence is given in the proof of Theorem 1 in [4] and for the reader's convenience, we sketch it. Assume that a sequence  $f_0, f_1, \dots, f_p$ , is defined so that  $f_0 = f$  and

$$f_j \in D(G_b^j) \cap \mathcal{H}_b^\perp \cap (D_-^b)^\perp, \quad j = 1, \dots, p.$$

Set

$$f_{p+1} := \frac{1}{\epsilon_{p+1}} \int_0^{\epsilon_{p+1}} V(\tau) f_p \, d\tau.$$

Then  $f_{p+1} \in D(G_b^{p+1})$  and

$$G_b f_{p+1} = \frac{1}{\epsilon_{p+1}} \left( V(\epsilon_{p+1}) f_p - f_p \right).$$

Next choose  $\epsilon_{p+1} > 0$  so that  $\epsilon_{p+1} < \epsilon/2$  and

$$\|G_b^k(f_{p+1} - f_p)\| \leq \frac{\epsilon}{2^{p+1}} \quad \text{for } k = 0, 1, \dots, p.$$

This is possible since  $V(t)$  is strongly continuous at 0 and

$$\|G_b^k(f_{p+1} - f_p)\| \leq \max_{0 \leq \tau \leq \epsilon_{p+1}} \|V(\tau) G_b^k f_p - G_b^k f_p\|.$$

Now for fixed integers  $N \geq 0$ ,  $p \geq N$ ,  $\mu \geq 0$  we get  $\|G_b^N(f_{p+\mu} - f_\mu)\| < \epsilon/2^p$  and since the operators  $G_b^p$  are closed, we can find  $\varphi_\epsilon \in \bigcap_{j=1}^\infty D(G_b^j) \cap \mathcal{H}_b^\perp \cap (D_-^b)^\perp$  so that  $\lim_{p \rightarrow \infty, p \geq N} G_b^N f_p = G_b^N \varphi_\epsilon$ ,  $\forall N \geq 0$ . Finally,  $\lim_{p \rightarrow \infty} f_p = \varphi_\epsilon$ , implies that  $\lim_{t \rightarrow \infty} V(t)\varphi_\epsilon = 0$ .

Applying the above argument, we conclude that  $V(t)\varphi_\epsilon = 0$  for  $t \geq T_0$  and passing to limit  $\epsilon \rightarrow 0$ , we deduce  $V(t)f = 0$  for  $t \geq T_0$ . This completes the proof of Theorem 4.2.  $\square$

**Corollary 4.5.** *With the assumptions of Theorem 4.2, the operator  $G_b$  has no outgoing eigenfunctions in  $\mathcal{H}_b^\perp$  and  $G_b^*$  has no incoming eigenfunctions in  $\mathcal{H}_b^\perp$ .*

*Proof.* If  $G_b f = \lambda f$  with  $f \perp D_-^a$  and  $u(t, x) = V(t)f = e^{\lambda t} f$ ,  $\text{Re } \lambda < 0$ , Theorem 4.2 says that  $u(t, x)$  must be disappearing and this yields  $f = 0$ . Notice that we can apply only a part of the argument of the proof of Theorem 4.2 leading to  $f = 0$  for  $|x| \geq b \geq 2\rho$ . Since  $Qf = 0$ , we have  $(G^{4d} + Q^2 - \lambda^{4d})f = 0$ . The symbol  $A(\xi)^{4d} + Q^2(\xi)$  of  $G^{4d} + Q^2$  is elliptic with constant coefficients so  $f = 0$  in  $\Omega$ .

For the operator  $G_b^*$  a similar argument shows that  $G_b^*$  has no incoming eigenfunctions in  $\mathcal{H}_b^\perp$ .  $\square$

To examine the completeness of the wave operators  $W_\pm$ , notice that (see [2])

$$\overline{\text{Ran } W_\pm} = \mathcal{H}_b^\perp \ominus \mathcal{H}_\pm,$$

where

$$\mathcal{H}_+ = \{f \in H : \lim_{t \rightarrow +\infty} V(t)f = 0\}, \quad \mathcal{H}_- = \{f \in H : \lim_{t \rightarrow +\infty} V^*(t)f = 0\}.$$

Next prove that  $\mathcal{H}_\pm \perp D_\pm^a$ ,  $a \geq \rho$ . In fact, if  $g \in D_-^a$ , then  $V(t)U_0(-t)g = g$  and for  $f \in \mathcal{H}_-$ ,

$$(f, g) = (f, V(t)U_0(-t)g) = (V^*(t)f, U_0(-t)g) \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

A similar argument works for  $\mathcal{H}_+$ .

**Example 4.6.** If  $\mathcal{H}_- = \mathcal{H}_+$ , and  $f$  is an eigenfunction of  $G_b$  with eigenvalue in  $\{\operatorname{Re} \lambda < 0\}$ , then the function  $f$  is outgoing and it follows that  $f = 0$ . Thus the existence of *at least one eigenvalue* of  $G_b$  in  $\operatorname{Re} z < 0$  implies that the wave operators  $W_\pm$  are not complete. This improves the result in [14].

**Example 4.7.** The above argument shows that if  $f$  is an eigenfunction for which  $G_b f = \lambda f$  with  $\operatorname{Re} \lambda < 0$  the outgoing component of  $f$  vanishes, so  $f$  is always incoming. Our explicit construction of such an eigenvalue for the Maxwell system in [1] with strictly dissipative boundary condition

$$(4.3) \quad E_{tan}(1 + \epsilon) - \nu \wedge B_{tan} = 0 \quad \text{on } |x| = 1, \quad \epsilon > 0$$

illustrates this situation.

*Remark 4.8.* The space  $\mathcal{H}_b^\perp \ominus D_-^a$  is invariant with respect to  $V(t)$ . An application of Theorem 4.2 proves that the semigroup  $V(t)$  restricted to this space has no eigenvalues in  $\operatorname{Re} z < 0$ . It is an open problem to find dissipative boundary conditions such that  $V(t)$  has no eigenvalues in  $\mathcal{H}_b^\perp$ .

Thanks to Corollary 4.5, we may apply the results of Section 5 in [10]. For the reader's convenience we use the same notations as in [10]. Let  $\rho_0(B)$  be the component of the resolvent set of  $B$  containing right half plane. Then Theorem 4.2 in [10] says that  $\mathcal{S}(z)$  can be continued analytically from the lower half plane into  $-\mathbf{i}\rho_0(B)$ . Thus  $\mathcal{S}(z)$  is meromorphic in  $\mathbb{C}$ . The poles of the scattering matrix in  $\operatorname{Im} z > 0$  are called scattering resonances and they form a discrete set. The crucial point of our argument is the following.

**Definition 4.9.**  $\hat{z}$  is a **zero of  $\mathcal{S}(z)$**  if there exists  $h \neq 0$  such that  $\mathcal{S}(\hat{z})h = 0$ .

**Theorem 4.10** (Theorem 5.6 in [10]). *If  $G_b|_{\mathcal{H}_b^\perp}$  has no outgoing eigenfunctions and  $G_b^*|_{\mathcal{H}_b^\perp}$  has no incoming eigenfunctions, then the point spectrum of  $G_b|_{\mathcal{H}_b^\perp}$  is of the form  $\mathbf{i}z$ , where  $\operatorname{Im} z > 0$  and  $z$  is a zero of  $\mathcal{S}(z)$  or possibly a resonance.*

*Proof of Theorem 1.4.* For  $\operatorname{Re} z < 0$  we know that the index of  $G_b - z$  is finite. To prove that this index is 0, it suffices to find at least one point  $\hat{z}$ ,  $\operatorname{Re} \hat{z} < 0$ , such that  $G_b - \hat{z}$  is invertible. Equation (3.4) and the argument of Section 3 show that  $\|K^\#\| < 1$  on a neighborhood of 0, so  $\mathcal{S}(z)$  has no zeros in that neighborhood. Since  $B$  has no spectrum on  $\mathbf{i}\mathbb{R}$ , there are no resonances on  $\mathbb{R}$ . By Theorem 4.8 there exists a neighborhood  $\mathcal{U}$  of 0, such that for every  $z \in \mathcal{U} = \{z \in \mathbb{C} : |z| < \epsilon_0, \operatorname{Im} z > 0\}$  the point  $\mathbf{i}z$  is not an eigenvalue of  $G_b$ .

An analogous argument works for  $G_b^*$ . First define a scattering operator  $S_1 = W_1 \circ W_+$  related to  $V^*(t)$  and  $U_0(t)$ . Then introduce a scattering matrix  $\mathcal{S}_1(z)$  and obtain an analogue of Theorem 4.2. Conclude that there exists a small neighborhood  $\mathcal{U}_1$  of 0 such that for  $z \in \mathcal{U}_1 = \{z \in \mathbb{C} : |z| < \epsilon_1, \operatorname{Im} z > 0\}$  the point  $\mathbf{i}z$  is not an eigenvalue of  $G_b^*$ . Consequently, for  $z \in \mathcal{U} \cap \mathcal{U}_1$  the index of  $G_b - \mathbf{i}z$  is 0 and we



can apply Theorem 2.4. This proves that outside a discrete set in  $\operatorname{Re} z < 0$  the operator  $G_b - z$  is invertible. If  $z$ ,  $\operatorname{Re} z < 0$ , is not an eigenvalue of  $G_b$ , but  $z \in \sigma(G_b)$ , then  $z$  must be in the residual spectrum of  $G_b$ . Then  $\dim \operatorname{Ker} (G_b - z) = 0$  and  $\operatorname{codim} \operatorname{Ran} (G_b - z) > 0$ . This leads to  $\operatorname{index} (G_b - z) > 0$  and we obtain a contradiction. The spectrum of  $G_b$  in  $\operatorname{Re} z < 0$  is formed only by isolated eigenvalues with finite multiplicities.

Finally, the space  $D_+^\rho$  is invariant with respect to the semigroup  $V(t)$ . Thus the generator  $G_b$  is the extension of the generator  $G_+$  of the semigroup  $V(t)|_{D_+^\rho} = U_0(t)|_{D_+^\rho}$ . By using the translation representation  $\mathcal{R}_n$ , it is easy to see that  $G_+$  has spectrum in  $\{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$ , so by a well known result the boundary  $i\mathbb{R}$  of  $\operatorname{Re} z < 0$  must be included in  $\sigma(G_b)$ . Since  $G_b$  over  $\mathcal{H}_b^\perp$  has no eigenvalues on  $i\mathbb{R}$ , we deduce that  $i\mathbb{R}$  is included in the continuous spectrum of  $G_b$ . This completes the proof of Theorem 1.4.  $\square$

## 5. PERTURBATIONS

In [1] we constructed an example of an eigenvalue of  $G_b$  in  $\mathcal{H}_b^\perp$  for Maxwell's equations on the exterior of a ball and with strictly dissipative boundary conditions. The construction relied in an essential way on spherical symmetry. One could imagine that these eigenvalues are very sensitive and would disappear under small perturbations of the boundary or the boundary conditions. In this section we show that in fact the eigenvalues are stable thus extending the construction to the non symmetric case.

Restrict attention to the Maxwells equations in  $\Omega \subset \mathbb{R}^3$ . Let  $\Lambda(x)$  be a smooth  $(2 \times 6)$  matrix-valued function defined for  $x \in \partial\Omega$  such that  $\operatorname{rank} \Lambda(x) = 2$ ,  $\Lambda(x)u = 0 \Leftrightarrow u \in \mathcal{N}(x)$ . Assume that the boundary condition  $\Lambda(x) = 0$  is strictly maximal dissipative, that is,  $\langle A(\nu(x))u(x), u(x) \rangle = 0$  for  $x \in \partial\Omega$  implies  $u(x) \in \operatorname{Ker} A(\nu(x))$ .

It is easy to see that  $(\operatorname{Ker} G_b)^\perp \subset \{u \in H^1(\Omega) : \operatorname{div} u = 0\}$ . For maximally strictly dissipative boundary conditions and  $u \in H^1(\Omega)$  we have the following coercive estimate stronger than (H)

$$(5.1) \quad \|u\|_{H^1(\Omega)} \leq C \left( \|(G - z)u\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{L^2(\Omega)} + \|\Lambda u\|_{H^{1/2}(\partial\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

with a constant  $C > 0$  depending on  $\Omega$ ,  $\Lambda$  and  $z$ . This estimate is a consequence of the fact that we can associate to our problem a boundary problem for a second order elliptic system which satisfies the Lopatinski condition (see, for instance, Theorem 2 and Section 5 in [7]).

The perturbation argument needs the following characterization of eigenvalues. When (5.3) is satisfied, it is also satisfied after small perturbations of the operator and the boundary condition. Combined with changes of variables close to the identity it also applies to small perturbations of the domain  $\Omega$ .

**Theorem 5.1.** *For a fixed  $z \in \mathbb{C}$ ,  $\operatorname{Re} z < 0$ , and strictly dissipative boundary condition  $\Lambda(x)u = 0$  the following conditions are equivalent:*

- (i)  $G_b - z$  is invertible.

(ii) There is a constant  $c > 0$  so that for all  $u \in D(G_b) \cap (\text{Ker } G_b)^\perp$  we have

$$(5.2) \quad c\|u\|_{H^1(\Omega)} \leq \|(G_b - z)u\|_{L^2(\Omega)}.$$

(iii) There is a constant  $c > 0$  so that for all  $u \in H^1(\Omega)$  we have

$$(5.3) \quad c\|u\|_{H^1(\Omega)} \leq \|(G - z)u\|_{L^2(\Omega)} + \|\text{div } u\|_{L^2(\Omega)} + \|\Lambda u\|_{H^{1/2}(\partial\Omega)}.$$

*Proof.* (i)  $\Leftrightarrow$  (ii). Since  $G_b - z$  is Fredholm of index zero, (i) holds if and only if there is a constant  $c > 0$  so that

$$c\|u\|_{D(G_b)} \leq \|(G_b - z)u\|_{L^2(\Omega)},$$

where  $\|u\|_{D(G_b)} = \left( \|u\|_{L^2(\Omega)}^2 + \|Gu\|_{L^2(\Omega)}^2 \right)^{1/2}$  is the graph norm.

The equivalence of (i) and (ii) then follows from the fact that the graph norm and the  $H^1(\Omega)$  norm are equivalent on  $D(G_b) \cap (\text{Ker } G_b)^\perp$  and that in turn follows from the condition (H).

(iii)  $\Rightarrow$  (ii). This is immediate since for  $u \in D(G_b) \cap (\text{Ker } G_b)^\perp$  we have  $\Lambda u = 0$  and  $\text{div } u = 0$ .

(i)  $\Rightarrow$  (iii). Assume that (iii) is violated. Then since (5.1) is satisfied, one can choose a sequence  $\{u^n\}$  bounded in  $H^1(\Omega)$  with

$$\|u^n\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \|(G - z)u^n\|_{L^2(\Omega)} + \|\text{div } u^n\|_{L^2(\Omega)} + \|\Lambda u^n\|_{H^{1/2}(\partial\Omega)} \rightarrow 0.$$

Passing to a subsequence, which we denote again by  $\{u^n\}$ , we may suppose that  $\{u^n\}$  converges weakly in  $H^1(\Omega)$  to a limit  $w \in H^1(\Omega)$ .

The key argument is to show that  $\{u^n\}$  is precompact in  $L^2(\Omega)$ . Assuming that, one concludes that there exists a subsequence  $\{u^{n_k}\}$  that converges strongly in  $L^2(\Omega)$  so  $\|w\|_{L^2} = 1$ .

The weak limits imply that  $(G - z)w = 0$ ,  $\text{div } w = 0$ , and  $\Lambda w = 0$  so  $w \neq 0$  is an element of the kernel of  $G_b - z$ . This contradicts (i).

It remains to prove the precompactness of  $\{u^n\}$  in  $L^2(\Omega)$ . Choose  $\phi \in C^\infty(\mathbb{R}^3)$  so that  $\phi = 0$  on a neighborhood of  $\bar{\Omega}$  and  $\phi = 1$  for  $|x| \geq R$  for some  $0 < R < \infty$ . Since  $\{u^n\}$  is bounded in  $H^1(\Omega)$  and  $(1 - \phi)$  has compact support, it follows that  $(1 - \phi)u_n$  is precompact in  $L^2(\mathbb{R}^3)$ . It remains to prove precompactness of  $\phi u^n$ . Define  $u^n := (E^n, B^n)$  and

$$zE^n - \text{curl } B^n := f_1^n, \quad zB^n + \text{curl } E^n := f_2^n, \quad \text{div } E^n := g_1^n, \quad \text{div } B^n := g_2^n.$$

Setting  $f^n = (f_1^n, f_2^n)$ ,  $g^n = (g_1^n, g_2^n)$ , by construction we have

$$\lim_{n \rightarrow \infty} \|f^n\|_{L^2(\Omega)} = \lim_{n \rightarrow \infty} \|g^n\|_{L^2(\Omega)} = 0.$$

Taking the divergence of the equation  $(z - G) \begin{pmatrix} E_n \\ B_n \end{pmatrix} = \begin{pmatrix} f_1^n \\ f_2^n \end{pmatrix}$  shows that

$$zg_1^n = z \text{div } E^n = \text{div } f_1^n, \quad zg_2^n = z \text{div } B^n = \text{div } f_2^n,$$

so

$$\lim_{n \rightarrow \infty} \|\text{div } f^n\|_{L^2(\Omega)} = 0.$$

Furthermore,

$$\begin{aligned}
 (5.4) \quad & z\phi E^n - \operatorname{curl} \phi B^n = \phi f_1^n - [\operatorname{curl}, \phi] B^n := \tilde{f}_1^n, \\
 & z\phi B^n + \operatorname{curl} \phi E^n = \phi f_2^n + [\operatorname{curl}, \phi] E^n := \tilde{f}_2^n, \\
 & \operatorname{div} \phi E^n = \phi g_1^n + [\operatorname{div}, \phi] E^n := \tilde{g}_1^n, \\
 & \operatorname{div} \phi B^n = \phi g_2^n + [\operatorname{div}, \phi] B^n := \tilde{g}_2^n.
 \end{aligned}$$

Since  $f^n$  and  $g^n$  tend to zero in  $L^2(\Omega)$ , and  $E^n$  and  $B^n$  are bounded in  $H^1(\Omega)$ , it follows from the expressions defining them that  $\tilde{f}^n$  and  $\tilde{g}^n$  are precompact in  $L^2(\mathbb{R}^3)$ .

We prove the precompactness of  $\{\phi E^n\}$ . The case of  $\phi B^n$  is entirely analogous. Compute

$$\begin{aligned}
 (5.5) \quad & z^2 \phi E^n = \operatorname{curl} z \phi B^n + z \tilde{f}_1^n = -\operatorname{curl} \operatorname{curl} (\phi E^n) + \operatorname{curl} \tilde{f}_2^n + z \tilde{f}_1^n \\
 & = \Delta(\phi E^n) + \operatorname{grad} \operatorname{div} (\phi E^n) + \operatorname{curl} \tilde{f}_2^n + z \tilde{f}_1^n.
 \end{aligned}$$

Therefore

$$(z^2 - \Delta)(\phi E^n) = \operatorname{grad} \tilde{g}_1^n + \operatorname{curl} \tilde{f}_2^n + z \tilde{f}_1^n.$$

Inverting  $z^2 - \Delta$  on tempered distributions on  $\mathbb{R}^3$  by using the fundamental solution, yields

$$\phi E^n = \frac{e^{z|x|}}{4\pi|x|} * \left( \operatorname{grad} \tilde{g}_1^n + \operatorname{curl} \tilde{f}_2^n + z \tilde{f}_1^n \right).$$

The expression in parentheses is precompact in  $H^{-1}(\mathbb{R}^3)$ . It follows that  $\phi E^n$  is precompact in  $H^1(\mathbb{R}^3)$  and therefore in  $L^2(\Omega)$ . This completes the proof.  $\square$

*Remark 5.2.* We can generalize Theorem 5.1 for systems with strictly dissipative boundary conditions satisfying the condition (E) for which we have an analogue of (5.1). Then  $\operatorname{div} u = 0$  is replaced by  $Qu = 0$  and for the proof we may exploit the fundamental solution of  $z^2 - G^2 + Q^*Q$ .

It follows from the characterization (5.3) that if  $G_b - z$  is invertible, then the same is true after small perturbations of  $G$ ,  $\Lambda$ ,  $z$  and the domain  $\Omega$ . If  $|z - \hat{z}| = r$  lies in  $\operatorname{Re} z < 0$  and there are no eigenvalues on this circle, the same is true after small perturbation, and the spectral projector

$$\frac{1}{2\pi i} \oint_{|z - \hat{z}| = r} (z - G_b)^{-1} dz$$

also depends continuously on perturbations. After perturbation, an eigenvalue may split into a finite number of points (no larger than the finite rank of the associated spectral projector).

#### REFERENCES

- [1] F. Colombini, V. Petkov and J. Rauch, *Incoming and disappearing solutions of Maxwell's equations*, Proc. AMS, **139** (2011), 2163-2173.
- [2] V. Georgiev, *Existence and completeness of the wave operators for dissipative hyperbolic systems*, J. Oper. Theory, **14** (1985), 291-310.
- [3] V. Georgiev, *Controllability of the scattering operator for dissipative hyperbolic systems*, Math. Nachr. **122** (1985), 339-346.
- [4] V. Georgiev, *Disappearing solutions for dissipative hyperbolic systems of constant multiplicity*, Hokkaido Math. J., **15** (1986), 357-385.

- [5] V. Georgiev and P. Stefanov, *Existence of the scattering operator for dissipative hyperbolic systems of variable multiplicity*, J. Oper. Theory, **19** (1988), 217-241.
- [6] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, I, Springer, Berlin, 1983.
- [7] A. Majda, *Coercive inequalities for non elliptic symmetric systems*, Comm. Pure Appl. Math. **28** (1975), 49-89.
- [8] A. Majda, *Disappearing solutions for the dissipative wave equation*, Indiana Math. J. **24** (1975), 1119-1133.
- [9] P. D. Lax and R. S. Phillips, *Scattering theory*, 2nd Edition, Academic Press, New York, 1989.
- [10] P. D. Lax and R. S. Phillips, *Scattering theory for dissipative hyperbolic systems*, J. Funct. Anal. **14** (1973), 172-235.
- [11] A. Majda and M. Taylor, *Inverse scattering problems for transparent obstacles, electromagnetic waves and hyperbolic systems*, Comm. in PDE, **2** (1977), 395-438.
- [12] V. Petkov, *Representation of the scattering operator for dissipative hyperbolic systems*, Comm. in PDE, **6** (1981), 993-1022.
- [13] V. Petkov, *Scattering theory for hyperbolic operators*, North Holland, 1989.
- [14] V. Petkov, *Scattering problems for symmetric systems with dissipative boundary conditions*, pp. 337-353 in *Studies in Phase Space Analysis with Applications to PDEs*, Progress in Non-linear Differential Equations and their Applications, **84**, Birkhäuser, 2013.
- [15] V. Petkov and L. Stoyanov, *Sojourn times of trapping rays and the behavior of the modified resolvent of the Laplacian*, Ann. Inst. H. Poincaré (Physique théorique) **62** (1995), 17-45.
- [16] J. Rauch, *Partial Differential Equations*, Graduate Texts in Mathematics 128, Springer-Verlag, 1992.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA, ITALIA

*E-mail address:* colombini@dm.unipi.it

INSTITUT DE MATHÉMATIQUES DE BORDEAUX, 351, COURS DE LA LIBÉRATION, 33405 TALENCE, FRANCE

*E-mail address:* petkov@math.u-bordeaux1.fr

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, USA

*E-mail address:* rauch@umich.edu